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VISCOUS EFFECT ON SURFACE WAVES GENERATED BY STEADY DISTURBANCES

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Abstract

A linearized theory is applied here to investigate the viscous effect on water waves generated and maintained by a system of external disturbances which is distributed over the free surface of an otherwise uniform flow. The flow is taken to be in the steady state configuration. The analysis is carried out to yield the asymptotic expressions for the surface wave when the Reynolds number of the flow is either large or small.

1. Introduction

The problem of the decay of a train of free-running simple waves over a water surface due to the viscous effect has been treated by Lamb[1] and Basset[2]. Since in the general practice water waves are usually generated by some localized external disturbances, this consideration suggested to the present authors the investigation of the viscous attenuation, in space, of the water waves generated and maintained by a system of external normal stress and shearing stress distributed over the free surface of an otherwise uniform flow. The scope of the present investigation will be limited to the case when the external forcing functions are independent of the time and the motion has been maintained for a long duration such that only the steady state solution of the problem will be of interest. Since the viscous attenuation turns out to depend significantly on the wave length, especially when the wave length is small, both the gravitational and surface tension effect are here taken into account.

It is well known that by neglecting the viscous effect, one is led in general to the so-called singular perturbation problem; because in doing so, the order of the differential equation governing the flow momentum is reduced by one and consequently certain boundary condition or conditions have to be relaxed. In the case of a fluid flow passing over a solid body, the condition that the velocity vanishes at the solid surface in the presence of viscosity is replaced by one that the velocity is tangential to the surface when the viscosity is neglected. In the case of a free surface flow, such as the present problem in question, the condition that states the balance of the viscous part of the total stress at the free surface must be omitted with the omission of the viscosity. One of the direct consequences that arises in such problems of steady, inviscid water waves is that the mathematical solution becomes, strictly speaking, indeterminate. More precisely, it is found to be not possible in general to obtain the unique mathematical solution with the desired physical properties by imposing only the boundedness conditions at infinity. Within the framework of the potential theory for the steady surface wave problems, some stronger radiation conditions are actually necessary; for example, in the absence of the surface tension effect, the steady surface wave is required to exist only on the downstream side of the external disturbance. The necessity of imposing such radiation conditions has been a topic of some dispute. To avoid such conditions, Rayleigh[3] introduced a small dissipative force, proportional to the relative flow velocity; thereby the irrotationality property of the flow is preserved. Recently, it has been pointed out by different authors (Green[4], Stoker[5], De Prima and Wu[6]) that the steady state solution can be obtained uniquely if the solution is interpreted as the limit of a corresponding initial value problem, as the time tends to infinity. Incidently, it was revealed in Ref. 6 that the coefficient of Rayleigh's force is actually a time-limiting factor. Now, with the additional viscous effect taken into account, it is shown in the present work that the mathematical solution becomes determinate, and bears automatically the desired physical properties (as required by the radiation conditions) and also exhibits the actual attenuation that takes place.

The outline of this paper is as follows. By assuming that the resulting motion is a small perturbation of the basic uniform flow, the same linearized theory as used by Lamb[1] is applied to treat the present problem. Oseen's equation is used to describe the momentum conservation; and the flow is further decomposed into two parts, one being irrotational and the other solenoidal. After the integral representation of the surface wave is obtained, by using the Fourier transform method, the analysis is carried out further to yield the asymptotic solutions in closed form for two special cases when the Reynolds number is large or small. The Reynolds number of this problem is defined as $Re = (U/\nu)(\sigma/g)^{1/2}$ where U is the free stream velocity, ν the kinematic viscosity of the fluid, g the gravitational constant and σ the ratio of the surface tension to the fluid density. For large values of Re (small viscous effect) and $U > c_m$ ($= (4g\sigma)^{1/4}$), the result shows that a train of gravity waves exists downstream, and capillary waves upstream of the disturbance. Both of these waves are attenuated by the viscosity; the smaller the wave length, the faster is the space rate of attenuation. In the critical case of $U = c_m$, the viscosity is found to play a significant role to assure the existence of the steady solution. For small values of Re (large viscous effect) and moderate values of U, it is shown that a train of viscous surface waves of large wave length propagates only on the downstream side, with its amplitude and wave length greatly affected by the viscosity.

2. Formulation of the Problem

The problem in question concerns the water waves generated by a pressure distribution and a viscous shearing stress applied on the free surface of a deep water which has a uniform free stream velocity U. We shall restrict ourselves to the two-dimensional motion in an xy-plane, with the y-axis pointing vertically upward and the x-axis coinciding with the undisturbed free surface, directed along U. The liquid medium is taken to be incompressible, of constant density ρ, and viscous, of kinematic viscosity ν. The resulting motion is assumed to be a small perturbation so that a linear theory can be applied. Furthermore, we are interested only in the steady state solution; the time-dependence of the flow will thereby be omitted. It will be seen later that, in contrast with the nonviscous case, the viscous steady state problem is uniquely determinate and hence no radiation condition need be imposed.

With the total flow velocity denoted by $U + \vec{q} = (U+u, v)$, the perturbation velocity $\vec{q}$ of the incompressible flow satisfies the continuity equation

$$\operatorname{div} \vec{q} = u_x + v_y = 0, \qquad |x| < \infty, \quad y < 0, \tag{1}$$

where the subscripts denote the partial derivatives. By assuming that $|\vec{q}| \ll U$, the Navier-Stokes equations for the steady flow in the region $|x| < \infty$, $y < 0$, may be linearized to give the Oseen equations

$$U \cdot u_x = -\frac{1}{\rho} p_x + \nu \nabla^2 u \,, \tag{2}$$

$$U v_x = -\frac{1}{\rho} p_y - g + \nu \nabla^2 v \,, \tag{3}$$

where p is the pressure, g the gravitational constant, and ∇^2 the Laplacian. Equations (1) - (3) are the basic equations governing the flow motion. On the displaced free surface, $y = \zeta(x)$, the kinematic boundary condition that the flow velocity is tangential to the displacement $\zeta(x)$, may be expressed, after linearization, as

$$U \zeta_x = v(x, 0) \,. \tag{4}$$

When both the gravity and surface tension are considered, the linearized boundary conditions for the stresses at the free surface are (cf. Ref. 1) :

$$p(x, 0) - 2\mu v_y(x, 0) + \rho\sigma \zeta_{xx} = P(x) , \tag{5}$$

$$\mu \left[u_y(x, 0) + v_x(x, 0) \right] = Q(x) . \tag{6}$$

In the above, P denotes the external pressure (or the normal stress acting in the y-direction with the sign reversed, to be more general), Q the external shearing stress acting in the x-direction, $\mu = \rho\nu$, and $(\rho\sigma)$ is the surface tension of the liquid-air interface. Equation (5) expresses the condition that the normal stress component, $(2\mu v_y - p)$, has a jump across the free surface by an amount $\rho\sigma\zeta_{xx}$ due to the surface tension effect; while Eq. (6) states the condition that the shearing stress is continuous at the free surface. The arbitrary functions $P(x)$ and $Q(x)$ are assumed to be absolutely integrable:

$$\int_{-\infty}^{\infty} |P(x)| \, dx < \infty , \qquad \int_{-\infty}^{\infty} |Q(x)| \, dx < \infty . \tag{7}$$

We further require that

> ζ, ζ_x be absolutely integrable; and u, v be absolutely integrable with respect to x for any fixed $y \leq 0$. (8)

This condition is suggested by the argument that when P and Q are subject to condition (7), the resulting disturbance will be damped out at infinity by the viscosity. This completes the statement of the problem.

It is known[1,7] that the linearized flow $(\vec{q}, p)$ governed by Eqs. (1)-(3) can be uniquely decomposed into two parts, one is irrotational $(\vec{q}_1 , p)$ and the other is solenoidal $(\vec{q}_2 , 0)$,

$$\vec{q} = \vec{q}_1 + \vec{q}_2 , \qquad \text{curl } \vec{q}_1 = 0 , \qquad \text{div } \vec{q}_2 = 0 , \tag{9}$$

such that

$$U\,\vec{q}_{1_x} = -\,\mathrm{grad}\left(\frac{1}{\rho}\,p+gy\right), \qquad U\,\vec{q}_{2_x} = \nu\nabla^2\,\vec{q}_2\,. \tag{10}$$

It follows from (9) that $\vec{q}_1$ has a velocity potential φ ,

$$\vec{q}_1 = \mathrm{grad}\,\varphi = (\varphi_x, \varphi_y)\,, \tag{11}$$

and $\vec{q}_2$ in this two-dimensional flow is conveniently expressed in terms of a stream function ψ ,

$$\vec{q}_2 = (\psi_y, -\psi_x)\,. \tag{12}$$

Since $\vec{q}_1$ also satisfies (1), φ satisfies the Laplace equation

$$\varphi_{xx}+\varphi_{yy} = 0 \qquad \text{for} \quad |x| < \infty, \quad y<0\,. \tag{13}$$

Substituting (12) in the second equation of (10) and integrating, we obtain

$$\psi_{xx}+\psi_{yy} = \frac{U}{\nu}\,\psi_x \quad \text{for} \quad |x| < \infty, \quad y<0\,. \tag{14}$$

Furthermore, the first equation of (10) can be integrated to yield

$$p/\rho = -\,U\varphi_x - gy\,. \tag{15}$$

Now p may be eliminated from (5) and (15). The boundary conditions then become

$$U\,\zeta_x = \varphi_y - \psi_x \qquad \text{on} \quad y = 0\,; \tag{16}$$

$$U\varphi_x + g\zeta - \sigma\,\zeta_{xx} - 2\nu\,(\varphi_{xx}+\psi_{xy}) = -\,P(x)/\rho \qquad \text{on} \quad y = 0\,; \tag{17}$$

$$\nu\,(2\varphi_{xy}+\psi_{yy}-\psi_{xx}) = Q(x)/\rho \qquad \text{on} \quad y = 0\,. \tag{18}$$

Our problem now becomes to first solve for φ , ψ and ζ from (13) and (14) with conditions (16)-(18); the velocity $\vec{q}$ is then obtained from (9), (11) and (12), and the pressure p, from (15).

Next we introduce the Fourier transform of f(x) and its inversion, defined as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx , \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k)\, dk .$$

The existence of the Fourier transform of φ, ψ, ζ etc. is actually implied by conditions (7) and (8). The transform of (13) and (14) under condition (8) are

$$\tilde{\varphi}_{yy} - k^2 \tilde{\varphi} = 0 , \qquad \tilde{\psi}_{yy} - (k^2 + \frac{U}{\nu} ik) \tilde{\psi} = 0 ;$$

they have the solution of the form

$$\tilde{\varphi} = A(k)\, e^{|k|y} , \qquad \tilde{\psi} = B(k)\, e^{y(k^2 + iUk/\nu)^{1/2}} \quad \text{for } y \leq 0 \tag{19}$$

provided $(k^2 + iUk/\nu)^{1/2}$ is defined to have its real part positive for k real. The arbitrary functions A(k) and B(k) can be determined from the transform of Eqs. (16)-(18), that is, after elimination of ζ, from

$$\left[|k| (g + \sigma k^2) - U^2 k^2 + 2i\nu U k^3\right] A - ik \left[g + \sigma k^2 + 2i\nu U k \left(k^2 + \frac{iUk}{\nu}\right)^{1/2} \right] B =$$
$$= - \frac{1}{\rho} iUk\, \tilde{P} ,$$

$$2i|k| A + (2k + iU/\nu)\, B = \tilde{Q}/(\rho \nu k) .$$

With A and B determined from the above equations, $\tilde{\zeta}$ is then given by the transform of (16) ,

$$iUk\, \tilde{\zeta}(k) = |k|\, A - ikB .$$

Substituting the value of A and B in the above equation and applying the inverse transform, one obtains, after some manipulation, the following integral representation of $\zeta(x)$,

$$\zeta(x) = -\frac{1}{2\pi\rho}\int_{-\infty}^{\infty} e^{ikx}\,\frac{\tilde{P}(k)+U^{-1}\left[(2\nu k+iU)k/|k| - 2\nu\left(k^2+iUk/\nu\right)^{1/2}\right]\tilde{Q}(k)}{(g+\sigma k^2)-|k|\,(U-2i\nu k)^2-4\nu^2k^2\left(k^2+iUk/\nu\right)^{1/2}}\,dk. \tag{20}$$

In the remaining part of this paper, we shall confine ourselves primarily to the discussion of ζ .

First, it is convenient to define the fundamental solutions of $\zeta(x)$:

$$\zeta(x) = H_P(x) \quad \text{when} \quad P(x) = \tfrac{1}{2}\rho U^2\delta(x), \quad Q(x) = 0 \;; \tag{21}$$

$$\zeta(x) = H_Q(x) \quad \text{when} \quad P(x) = 0, \quad Q(x) = \tfrac{1}{2}\rho U^2\delta(x) ,$$

where $\delta(x)$ denotes the Dirac delta function. Thus, $H_P(x)$ and $H_Q(x)$ are the surface displacement due to respectively a concentrated normal pressure and a concentrated shearing stress at the origin. Since the problem is linear, the solution of $\zeta(x)$ for arbitrary $P(x)$ and $Q(x)$ can be deduced by superposition:

$$\zeta(x) = \left(\tfrac{1}{2}\rho U^2\right)^{-1}\int_{-\infty}^{\infty}\left\{H_P(x-\xi)\,P(\xi)+H_Q(x-\xi)\,Q(\xi)\right\}d\xi . \tag{22}$$

The integral representation of H_P is obtained simply by setting $\tilde{P}(k) = \frac{1}{2}\rho U^2$ and $\tilde{Q}(k) = 0$ in (20). In terms of the following nondimensional quantities,

$$x' = k_m x, \quad k' = k/k_m, \quad u_o = U/c_m, \quad \alpha = \nu k_m/U , \tag{23a}$$

where

$$k_m = (g/\sigma)^{1/2}, \qquad c_m = (4g\sigma)^{1/4}, \tag{23b}$$

$H_P(x)$ is expressed, after the prime is dropped for k', as

$$H_P(x) = -\frac{u_o^2}{\pi} \, R\ell \int_o^\infty \frac{\exp(i x' k)}{f(k; u_o, a)} \, dk , \tag{24}$$

with $f(k; u_o, a) = (k^2 + 1) - 2u_o^2 k(1 - 2i a k)^2 - 8u_o^2 a k^2 \left[a k (a k + i)\right]^{1/2}$. (25)

Similarly,

$$H_Q(x) = -\frac{u_o^2}{\pi} \, R\ell \int_o^\infty e^{ix'k} \, \frac{(i + 2a k) - 2\left[a k (a k + i)\right]^{1/2}}{f(k; u_o, a)} \, dk . \tag{26}$$

In (23), c_m is the minimum phase velocity of a train of simple harmonic surface waves in a nonviscous medium and k_m, the corresponding wave number; k_m^{-1} is thus a characteristic length of this problem. The quantity a may be regarded as the inverse of the Reynold number $Re = U/(\nu k_m)$; it provides a measure of the relative importance between the viscous effect and the inertia effect. When the viscosity is neglected, $a = 0$, the denominator of the integrand in (24) and (26) becomes

$$\begin{aligned} f(k; u_o, 0) &= k^2 - 2u_o^2 k + 1 = (k - \kappa_1)(k - \kappa_2) , \\ \kappa_1 &= u_o^2 - (u_o^4 - 1)^{1/2} , \qquad \kappa_2 = \kappa_1^{-1} , \end{aligned} \tag{27}$$

which has two simple zeros on the positive real k-axis at κ_1 and κ_2 for $u_o > 1$ (or $U > c_m$), or a double zero at $k = 1$ for $u_o = 1$, or two complex conjugate zeros for $u_o < 1$. Hence, with $a = 0$ and $u_o \geq 1$, the problem, strictly speaking, is indeterminate. The steady nonviscous problem may be made determinate either by imposing a proper radiation condition at $x = \pm\infty$, or by adopting some artifice such as that due to Rayleigh[3], or by considering anew the corresponding initial value problem[4, 5, 6]. However, with the viscosity of the medium taken into account, $\alpha > 0$, the integrand of (24) and (26) can be shown to have no singularity on the positive real k-axis (the path of integration). Since for $k > 0$, the imaginary part of $f(k; u_o, \alpha)$ can be written

$$\operatorname{Im} f(k; u_o, \alpha) = 8u_o^2 \alpha k^2 \left\{ 1 - \left(\frac{\alpha k}{2}\right)^{1/2} \left[(1 + \alpha^2 k^2)^{1/2} - \alpha k \right]^{1/2} \right\},$$

the quantity inside the curly bracket decreases monotonically from unity at $k = 0$ to $1/2$ at $k = \infty$. Furthermore, the real part of $f(k; u_o, \alpha)$ does not vanish at $k = 0$. This proves the above statement. Therefore, the consideration of the viscous effect not only will exhibit the physical attenuation of the surface wave that takes place, but also makes the problem mathematically determinate.

3. Analytic Behavior of the Integral Representation

In order to facilitate the evaluation of integrals (24) and (26), we investigate first the analytic behavior of their integrands with u_o and α regarded as two positive parameters. The following discussion will be limited to that of (24), while the discussion of (26) follows almost in parallel. The function $f(k; u_o, \alpha)$ of (25) will be abbreviated simply as $f(k)$ whenever it is unnecessary to mention its dependence on u_o or α.

If k is taken to be a complex variable, $f(k)$ has two branch points at $k = 0$ and $k = -i/\alpha$. With the introduction of a branch cut connecting these two points along the imaginary k-axis, $f(k)$ is then an analytic function of k in the cut plane. As the theorem of residues will be applied to evaluate integrals (24) and (26), we proceed to find the poles of the integrand, or the zeros of $f(k)$. We next show that $f(k)$ has exactly two simple zeros in the cut k-plane for all positive, real values of u_o and α, provided that 2α is different from $(u_o^4 + 1)^{1/2} \pm u_o^2$.

To show this, it is convenient first to remove the branch cut by the transformation

$$4\alpha k = (t - i)^2/t \tag{28}$$

which maps the entire cut k-plane conformally into the region $|t| \geq 1$ of the complex t-plane. Then

$$f(k(t)) = (16\alpha^2 t^3)^{-1} \left\{ t(t-i)^4 + 16\alpha^2 t^3 + 4i\alpha u_o^2 (t-i)^2 (t^3 + i t^2 + t - i) \right\} \quad . \tag{29}$$

It is thus obvious that f has a triple pole at $t = 0$ and five zeros in the entire t-plane. Now on the unit circle $|t| = 1$, let $t = e^{i\theta}$, then

$$4\alpha^2 f = 4\alpha^2 - (1 - \sin\theta)^2 (1 - 4\alpha u_o^2 \cos\theta) + 4i\alpha u_o^2 \sin^2\theta\, (1 - \sin\theta)^2. \tag{30}$$

Hence on $t = e^{i\theta}$, $f = 0$ has two particular solutions for $\alpha > 0$:

$$\theta = 0 \quad \text{with} \quad 4\alpha^2 + 4\alpha u_o^2 - 1 = 0 \qquad (\text{or} \;\; 2\alpha = (u_o^4 + 1)^{1/2} - u_o^2)\;; \tag{31a}$$

$$\theta = \pi \quad \text{with} \quad 4\alpha^2 - 4\alpha u_o^2 - 1 = 0 \qquad (\text{or} \;\; 2\alpha = (u_o^4 + 1)^{1/2} + u_o^2)\;. \tag{31b}$$

The relationships between α and u_o shown in the above parentheses are taken to keep α positive, real. As u_o increases from zero to infinity, α of (31a) decreases from $1/2$ to zero, whilst α of (31b) increases from $1/2$ to infinity. By expanding f near the point $t = \pm 1$, one finds that the above two particular solutions are two simple zeros of f. In the original cut k-plane, these two particular simple zeros of f lie on the two sides of the cut

$$\text{at} \quad k = -\frac{i}{2a} + 0 \qquad \text{with} \qquad 2a = (u_o^4 + 1)^{1/2} - u_o^2 \,, \tag{32a}$$

$$\text{and at} \quad k = -\frac{i}{2a} - 0 \qquad \text{with} \qquad 2a = (u_o^4 + 1)^{1/2} + u_o^2 \,. \tag{32b}$$

The physical significance of these particular solutions of $f=0$ is as yet not clear to the authors. Their contribution to the final solution, however, will be made explicit later for the special cases of interest.

In the general case when a and u_o are not restricted by the relations in (31), f then has no zero on the unit circle $|t| = 1$, or on the cut in the k-plane. Furthermore, Eq. (30) shows that on $t = e^{i\theta}$, f is a periodic function of θ with period 2π, and Im f is non-negative with $a > 0$. Hence

$$\frac{1}{2\pi i} \oint_{|t|=1} \frac{1}{f} \cdot \frac{df}{dt}\, dt = \frac{1}{2\pi} \arg f \Bigg|_{\theta=0}^{\theta=2\pi} = 0$$

which, from the theory of functions[8], shows that inside $|t| = 1$, the number of zeros of f is equal to the number of poles of f, a zero or a pole of order m being counted m times. But the only pole of f (see Eq. 29) is at $t = 0$ and is of order 3, hence the number of f inside $|t| = 1$ is also 3. Since f is known to have five zeros in the entire t-plane, this leaves f to have two zeros outside $|t| = 1$. Therefore, it follows from the conformal transformation that f has exactly two zeros in the entire cut k-plane provided a and u_o are not restricted by the relations in (31). This completes our proof.

Having known the number of zeros of $f(k)$, we can easily determine these zeros asymptotically for small or large values of α as follows.

3.1. Small values of α.

When α is taken to be small, the two zeros of $f(k)$, say k_1 and k_2, will lie near their corresponding nonviscous value κ_1 and κ_2 (see Eq. 27) because k_1 and k_2 are seen to be continuous functions of α. The expansion of $f(k; u_o, \alpha)$, obtained from (25), for α small suggests that k_1 and k_2 may be expanded as

$$k_1(u_o, \alpha) = \kappa_1 + a_1\alpha + b_1\alpha^{3/2} + c_1\alpha^2 + \dots ,$$

$$k_2(u_o, \alpha) = \kappa_2 + a_2\alpha + b_2\alpha^{3/2} + c_2\alpha^2 + \dots . \tag{33a}$$

Upon substitution of these expansions into that of $f(k) = 0$, we find

$$a_1 = \frac{4iu_o^2\kappa_1^2}{(u_o^4-1)^{1/2}}, \quad a_2 = -\frac{4iu_o^2\kappa_2^2}{(u_o^4-1)^{1/2}}, \quad b_1 = -e^{i\frac{\pi}{4}}\frac{4u_o^2\kappa_1^{5/2}}{(u_o^4-1)^{1/2}},$$

$$b_2 = e^{i\frac{\pi}{4}}\frac{4u_o^2\kappa_2^{5/2}}{(u_o^4-1)^{1/2}} \tag{33b}$$

where κ_1 and κ_2 are given by (27). Thus for $u_o > 1$ and α small, k_1 lies in the first quadrant whilst k_2, in the fourth quadrant of the k-plane. The above expansion also holds valid for $u_o < 1$; it breaks down, however, as u_o approaches unity. At $u_o = 1$, we have the different expansion:

$$k_1(1, \alpha) = 1 - 2(1-i)\alpha^{1/2} - i(8+\sqrt{2})\alpha + 0(\alpha^{3/2}),$$

$$k_2(1, \alpha) = 1 + 2(1-i)\alpha^{1/2} - i(8-\sqrt{2})\alpha + 0(\alpha^{3/2}), \tag{34}$$

so k_1 and k_2 still lie separately in the first and fourth quadrant of the k-plane.

For a given small α, there exists a large u_o such that the particular relation in (32a) is satisfied. In that case, f(k) has an additional zero at $k = -i(2\alpha)^{-1}$. But since this zero of f(k) lies on the imaginary axis at a large distance from the origin for $\alpha \ll 1$, its contribution to the motion can be seen to be rapidly damped out away from the origin and will thus be neglected in this case of small α.

3.2. A critical value of α.

Suppose that f(k) has a simple zero at $k=a+ib$ (and hence a simple pole of the integrand of Eq. 24). If the theorem of residues is applied with an appropriate construction of the integration contour to enclose this pole, the contribution from the residue at this pole will therefore contain a term $\exp(-|bx'| + iax)$ so that the real part a of the pole gives the wave number and the imaginary part b gives the attenuation of the surface wave. The surface ceases to have the wavy form when a vanishes; this condition will be referred to as the critical condition.

On the positive imaginary t-axis, we let $t = i\eta$, $\eta \geq 0$. Then from (29)

$$f = -(16\alpha^2\eta^3)^{-1}\left\{\eta\left[(\eta-1)^4 - 16\alpha^2\eta^2\right] + 4i\alpha u_o^2(\eta-1)^2(\eta^3+\eta^2-\eta+1)\right\} .$$

Now $\eta^3+\eta^2-\eta+1 = \eta^3+(\eta-1)^2+\eta > 0$ for $\eta \geq 0$, so that Im f = 0 at $\eta = 1$ only. But since $\mathcal{R}\ell\ f \neq 0$ at $\eta = 1$ with $\alpha > 0$, f has no zero on the positive imaginary t-axis. On the negative imaginary t-axis, let $t = -i\eta$, $\eta \geq 0$; then

$$f = -(16\alpha^2\eta^3)^{-1}\left\{\eta\left[(\eta+1)^4 - 16\alpha^2\eta^2\right] + 4i\alpha u_o^2(\eta+1)^2(\eta^3-\eta^2-\eta-1)\right\}.$$

The factor $(\eta^3-\eta^2-\eta-1)$ in the imaginary part of f can be shown to have only one real zero at

$$\eta_2 = \frac{1}{3}\left\{(19+\sqrt{297})^{1/3} + (19-\sqrt{297})^{1/3} + 1\right\} = 1.8393 ,$$

for which $R\ell$ f also vanishes provided

$$\alpha = (\eta_2+1)^2/4\eta_2 = 1.0957 .$$

We denote this zero of f by

$$t_2 = -(1.8393)i \qquad \text{with} \qquad \alpha = 1.0957 = \alpha_c \text{ say ,} \tag{35}$$

which is independent of the value of u_o. Transforming back to the k-plane by (28), we may then assert that f is free from zeros on the positive imaginary k-axis, and has only one zero on the negative imaginary k-axis at

$$k_2 = -i \quad \text{for} \quad \alpha = \alpha_c = 1.0957 \text{ and for all real } u_o > 0. \tag{36}$$

The above α_c is the only critical value of α for which the zero of f is purely imaginary.

3.3 Large values of α.

For $|t| > 1$ and $\alpha \gg 1$, the leading terms in the expansion of f obtained from (29) start with $f \cong 1 + i u_o^2 t^2 (4\alpha)^{-1}$. This suggests that the two zeros of f outside $|t| = 1$ may be expanded in the form

$$t_1 \cong \frac{2}{u_o}\alpha^{1/2} e^{i\pi/4} + a_3 + b_3\alpha^{-1/2} + c_3\alpha^{-1} + \dots , \tag{37a}$$

$$t_2 \cong -\frac{2}{u_o}\alpha^{1/2} e^{i\pi/4} + a_4 + b_4\alpha^{-1/2} + c_4\alpha^{-1} + \dots . \tag{37b}$$

Substituting these expressions in (29) and setting $f=0$, we find

$$a_3 = a_4 = \frac{i}{2}\,, \qquad b_3 = -b_4 = -\frac{4+9u_o^4}{16u_o^3}\, e^{-i\pi/4}\,, \qquad c_3 = c_4 = \frac{3+4u_o^4}{8u_o^2}\,. \tag{37c}$$

In order to have the above asymptotic expansions valid for large α and at the same time to retain $|t_1| > 1$, $|t_2| > 1$, u_o must be bounded away from zero and must not be large, or more precisely, $0(\alpha^{-1}) < u_o^2 < \alpha$. When $u_o^2 = 0(\alpha^{-1})$ and $\alpha \gg 1$, the first and the last term in the curly bracket of (29) must be considered together; in this manner one obtains

$$\left.\begin{matrix} t_1 \\ t_2 \end{matrix}\right\} = \pm \frac{4i\alpha}{(1+i\beta)^{1/2}} + \frac{2i(1+i\beta/4)}{1+i\beta} \pm \frac{1+i\beta+9\beta^2/8}{4i(1+i\beta)^{3/2}}\,\frac{1}{\alpha} + \frac{1-i\beta}{8i\alpha^2} + 0(\alpha^{-3})\,, \tag{38}$$

valid for $\beta \equiv 4\alpha u_o^2 = 0(1)$, $\alpha \gg 1$. On the other hand, when $u_o = 0(\alpha)$ and $\alpha \gg 1$, the last term in the curly bracket of (29) becomes predominant. Now this term has a double zero at $t=i$ and the factor (t^3+it^2+t-i) can be shown to have two zeros inside $|t| = 1$ and the third zero at $t=-(1.8393)i$ (see Eq. 35). By expanding f about $t=i$ and $-(1.8393)i$, one finds the two zeros of f outside $|t| = 1$ given by

$$\begin{aligned} t_1 &= i + \frac{1+i}{(\gamma^2\alpha)^{1/2}} + (\gamma^2\alpha)^{-1} - \frac{3}{4}(1-i)(\gamma^2\alpha)^{-3/2} + 0(\alpha^{-2})\,, \\ t_2 &= -(1.8393)i - 0.565(\gamma^2\alpha)^{-1} + i(0.033)(\gamma^2\alpha)^{-2} + 0(\alpha^{-3})\,, \end{aligned} \tag{39}$$

which hold valid for $\gamma \equiv u_o/\alpha$ of the order unity or greater, and $\alpha \gg 1$.

It may be noted that all the above different expansions become invalid for $u_o = 0(\alpha^{1/2})$. In this range of u_o the last two terms in the curly bracket of (29) are of the same order of magnitude and must be considered together; consequently the evaluation of the two zeros of f outside $|t| = 1$ becomes in this case so complicated that it does not seem worthwhile to get the explicit determination. However, since the two zeros of f, $t_1(u_o, \alpha)$ and $t_2(u_o, \alpha)$, are seen to be continuous functions of u_o and α, the qualitative behavior of t_1 and t_2 (and hence the behavior of the solution ζ) when $u_o = 0(\alpha^{1/2})$ and $\alpha \gg 1$ may always be obtained by "interpolating" their behavior at $u_o = 0(1)$ and $u_o = 0(\alpha)$. It may also be pointed out that in this range of $u_o = 0(\alpha^{1/2})$, a particular solution of $f = 0$ exists (see Eq. 31b):

$$t_3 = -1 \qquad \text{for} \qquad u_o^2 = \alpha + 0(\alpha^{-1}) \quad \text{and} \quad \alpha \gg 1 . \tag{40}$$

Finally, transforming the above results back to the original k-plane by (28), we obtain, for $\alpha \gg 1$, the zeros of f(k) given by:

$$\left.\begin{matrix} k_1 \\ k_2 \end{matrix}\right\} = \pm i(1+i\beta)^{-1/2} + \frac{3\beta}{8}(1+i\beta)^{-1}\alpha^{-1} + 0(\alpha^{-2})$$

$$= \pm(1+\beta^2)^{-1/4} \exp\left[i \tan^{-1} \frac{(1+\beta^2)^{1/2}+1}{\beta} \right] + \frac{3\beta}{8\alpha}(1+\beta^2)^{-1/2} e^{-i\tan^{-1}\beta}$$

$$+ 0(\alpha^{-2}) , \tag{41}$$

valid for $\beta \equiv 4\alpha u_o^2$ of the order unity or smaller;

$$\left.\begin{matrix} k_1 \\ k_2 \end{matrix}\right\} = \pm \frac{1}{2u_o} \alpha^{-1/2} e^{i\pi/4} - \frac{3i}{8\alpha} + 0(\alpha^{-3/2}) \qquad \text{for} \qquad u_o = 0(1) ; \tag{42}$$

and for $\gamma \equiv u_o/\alpha$ of the order unity or greater,

$$k_1 = \frac{1}{2\alpha^2} + \frac{i}{\gamma^2\alpha^3} + 0(\alpha^{-7/2}) , \quad k_2 = -i\,\frac{1.0957}{\alpha} - \frac{0.0995}{\gamma^2\alpha^2} + 0(\alpha^{-3}) . \tag{43}$$

As $U \to 0$, then $u_o \to 0$ and $\alpha \to \infty$ such that $\beta = 4\alpha u_o^2 \to 0$, hence from Eq. (41) $k_1 \to i$ and $k_2 \to -i$, which are the corresponding nonviscous solution. Thus, we can foresee that the viscous effect becomes insignificant at small values of U. Physically this is because the variation in velocity becomes everywhere too small for the viscosity to be effective.

By summing up the results (33), (34), (36), (41)-(43), we obtain the loci of the two zeros k_1 and k_2 of $f(k)$ for some specific ranges of u_o and α, as depicted in Fig. 1. The curves in the upper half plane are the loci of $k_1(u_o, \alpha)$ and those in the lower half plane, the loci of $k_2(u_o, \alpha)$. As $\alpha \to 0$, k_1 and k_2 approach their nonviscous value κ_1 and κ_2 given by (27). When α increases, k_1 has its real part decreasing and converges toward the origin (except when $U \to 0$), as $\alpha \to \infty$, the loci of k_1 being bounded within the first quadrant; whilst k_2 moves across the point $k = -i$ at $\alpha = 1.0957$ for all values of u_o into the third quadrant and then approaches the origin (again except when $U \to 0$) as $\alpha \to \infty$.

4. Solution for Small Values of α (Large Reynolds Number)

The integral (24) will now be calculated by applying the theorem of residues. For $x > 0$, we construct a closed contour Γ consisting of the original path along the real k-axis from $k = 0$ to $k = R$, a circular arc of large radius $|k| = R$ in the first quadrant and a return along the imaginary axis from $k = iR$ back to the origin. Then Γ encloses one simple pole at $k = k_1$, given by (33) or (34) for $0 < \alpha \ll 1$. By letting $R \to \infty$, the contribution along the circular arc vanishes. Hence, application of the

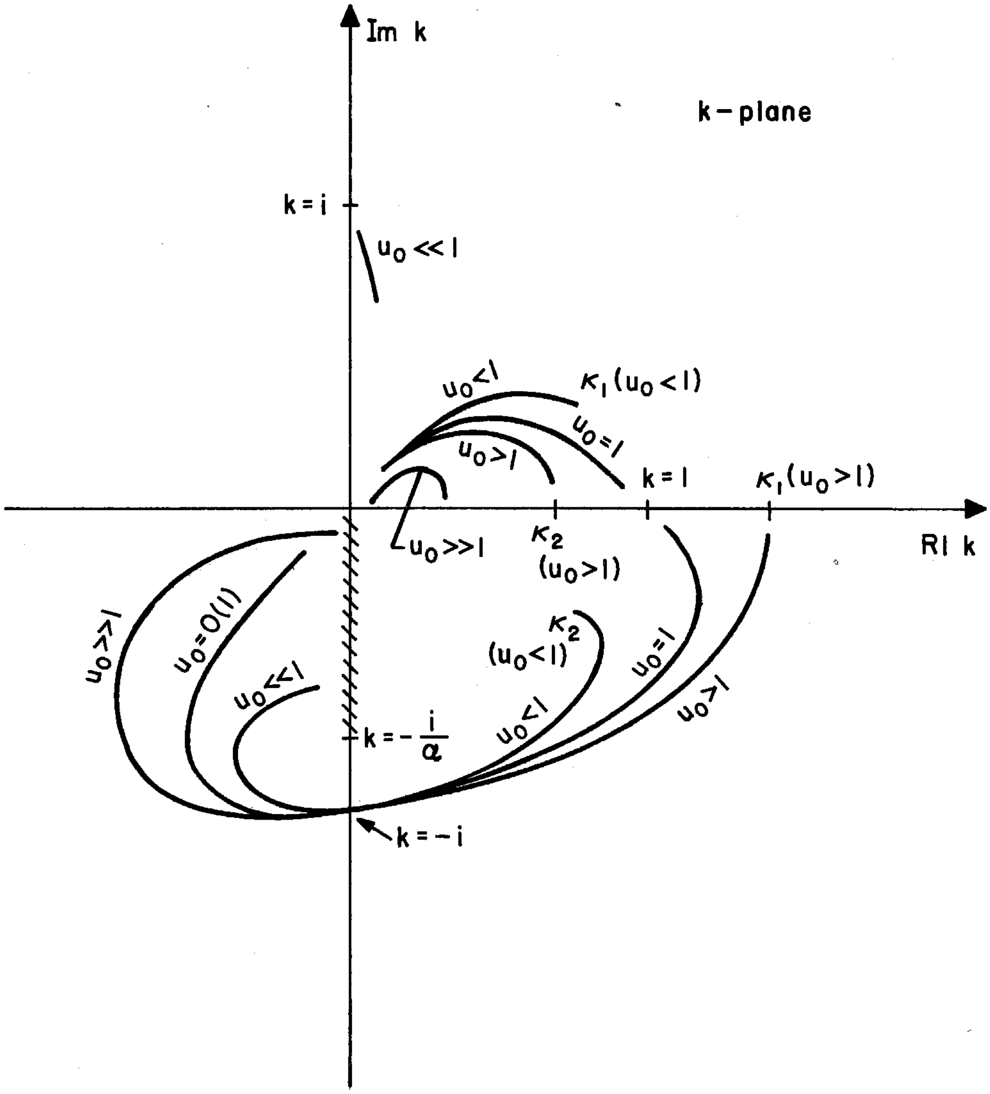
Im k
k-plane
k = i
$u_0 \ll 1$
$u_0 < 1$
$\kappa_1 (u_0 < 1)$
$u_0 = 1$
$u_0 > 1$
k = 1
$\kappa_1 (u_0 > 1)$
$u_0 \gg 1$
κ_2
$(u_0 > 1)$
Rl k
$u_0 \gg 1$
$u_0 = O(1)$
$u_0 \ll 1$
κ_2
$(u_0 < 1)$
$u_0 < 1$
$u_0 = 1$
$u_0 > 1$
$k = -\frac{i}{\alpha}$
k = -i

Figure 1.

theorem of residues to (24) yields, for $x > 0$,

$$H_p(x) = -(u_o^2/\pi)\, R\ell \left[2\pi i \text{ Res. } (k_1) \right] + L(x) , \tag{44a}$$

$$L(x) = -\frac{u_o^2}{\pi} R\ell \int_o^{i\infty} \frac{e^{ikx'}\, dk}{f(k; u_o\, a)} , \tag{44b}$$

where the integration is performed along the imaginary axis, and Res. (k_1) denotes the residue of the integrand at $k = k_1$, given by

$$\text{Res.}(k_1) = \lim_{k \to k_1} (k - k_1)\, e^{ikx'}/f(k) = e^{ik_1 x'} \left[df(k_1)/dk \right]^{-1} . \tag{44c}$$

The integral $L(x)$ converges even when $a = 0$, hence to obtain the first approximation of $L(x)$, the a appearing in its integrand may be neglected for $0 < a \ll 1$. Then $f(k; u_o, 0)$ may be factorized as (27). Hence with $k = iw$,

$$L(x) \cong \frac{u_o^2}{\pi} R\ell\; i \int_o^{\infty} \frac{e^{-x'w}\, dw}{(w + i\kappa_1)(w + i\kappa_2)} + 0(a)$$

$$\cong \frac{u_o^2}{\pi} R\ell \frac{1}{\kappa_2 - \kappa_1} \left\{ e^{i\kappa_1 x} \text{ei}\,(i\kappa_1 x) - e^{i\kappa_2 x} \text{ei}(i\kappa_2 x) \right\} + 0(a) \tag{45}$$

where κ_1, κ_2 are given by (27) and ei(z) denotes the exponential integral

$$\text{ei}(z) = \int_z^{\infty} e^{-t} \frac{dt}{t} . \tag{45a}$$

From the known asymptotic expansions of ei(z) for $|z|$ large or small[9], one readily deduces that for $|\kappa_1 x| \gg 1$,

$$L(x) \cong \frac{u_o^4}{\pi x^2} \left\{ 1 - 3! \frac{2(2u_o^4 - 1)}{\chi^2} + 0(|\kappa_1 x|^{-4}) \right. + 0(\alpha) \tag{46}$$

which holds valid for all u_o; and for $|\kappa_2 x| \ll 1$,

$$L(x) \cong \frac{u_o^2}{2\pi} (u_o^4 - 1)^{-1/2} \log \frac{\kappa_2}{\kappa_1} - \frac{1}{2} u_o^2 x + 0(x^2, \alpha) \qquad \text{for} \quad u_o > 1 ,$$

$$\cong \frac{u_o^2}{\pi} (1 - u_o^4)^{-1/2} \cos^{-1} u_o^2 - \frac{1}{2} u_o^2 x + 0(x^2, \alpha) \qquad \text{for} \quad u_o < 1 , \tag{47}$$

$$\cong \frac{1}{\pi} - \frac{1}{2} x + 0(x^2, \alpha) \qquad \text{for} \quad u_o = 1.$$

Since $L(x) \to 0$ like x^{-2} as $x \to +\infty$, $L(x)$ merely represents the local elevation which is important only in a neighborhood of the origin. By making use of (25), (33), (34) in calculating Res.(k_1), we finally obtain for $x > 0$, $0 < \alpha \ll 1$,

$$H_P(x) = - \frac{u_o^2}{(u_o^4 - 1)^{1/2}} \exp\left[- \frac{4\alpha u_o^2 \kappa_1^2 x'}{(u_o^4 - 1)^{1/4}} \right] \sin \kappa_1 x' + L(x') + 0(\alpha) \qquad \text{for} \quad u_o > 1 ,$$

$$= - \frac{u_o^2}{(1 - u_o^4)^{1/2}} \exp\left[- \left(\sqrt{1 - u_o^4} - 8 u_o^4 \alpha \right) x' \right] \cos u_o^2 x' + L(x') + 0(\alpha) \quad \text{for} \quad u_o < 1 , \tag{48}$$

$$= - \frac{1}{2(2\alpha)^{1/2}} e^{-2\sqrt{\alpha}\, x'} \cos\left[(1 - 2\sqrt{\alpha})\, x' - \frac{\pi}{4} \right] + L(x') + 0(\alpha) \qquad \text{for} \quad u_o = 1 ,$$

in which $L(x')$ is given by (45) and may be approximated as shown in (46), (47).

For $x < 0$, we take the closed contour Γ to be the boundary of the fourth quadrant of the k-plane, with the branch cut on the imaginary axis lying just outside of Γ. It was shown in the last section that for $0 < a \ll 1$, Γ encloses only one simple pole at $k = k_2$, given by (33) or (34). Hence, applying again the theorem of residues, we obtain

$$H_P(x) = -\frac{u_o^2}{\pi} R\ell \left\{ 2\pi i \frac{e^{ik_2 x'}}{df(k_2)/dk} + \int_o^{-i\infty} \frac{e^{ikx'}dk}{f(k; u_o, a)} \right\} \qquad \text{for} \quad x < 0 ,$$

where the integral is carried out along the negative imaginary axis to the right of the branch cut. We have shown that for $0 < a \ll 1$, the integrand has no pole on the imaginary axis except when u_o is so large that a and u_o satisfy (32a), then the integrand has a pole at $k = -i/2a = -2iu_o^2$ on the branch cut. In the latter case the path of integration is indented around this pole; the contribution of the integral on the indentation then yields a term proportional to $\exp(x'/2a)$ for $x < 0$, which is negligibly small since $a \ll 1$. With this contribution and the a in the integrand neglected, the result will then be accurate up to $0(a)$ (since this is the lowest order that a appears in the integrand). Carrying out the details as the previous case for $x > 0$, we obtain here for $x < 0$, $0 < a \ll 1$,

$$H_P(x) = -\frac{u_o^2}{(u_o^4 - 1)^{1/2}} \exp\left[\frac{4a u_o^2 \kappa_2^2 x'}{(u_o^4-1)^{1/2}} \right] \sin \kappa_2 x' + L(|x'|) + 0(a) \quad \text{for} \quad u_o > 1,$$

$$= -\frac{u_o^2}{(1-u_o^4)^{1/2}} \exp\left[\left(\sqrt{1-u_o^4} + 8a u_o^2\right) x' \right] \cos u_o^2 x' + L(|x'|) + 0(a) \text{ for } u_o < 1, \quad (49)$$

$$= -\frac{1}{2\sqrt{2a}} e^{2\sqrt{a}\, x'} \cos\left[(1 + 2\sqrt{a})\, x' - \frac{\pi}{4} \right] + L(|x'|) + 0(a) \qquad \text{for} \quad u_o = 1$$

where $L(|x'|)$ represents the same function defined by (45) with x now replaced by $|x|$.

Some important features of the viscous effect may be pointed out here. First, with nonvanishing viscosity the problem is mathematically determinate so that for $u_o > 1$ and α small, the gravity wave $\sin \kappa_1 x$ automatically appears only downstream $(x > 0)$ and the capillary wave $\sin \kappa_2 x$, only upstream. These waves are attenuated by the viscosity at the rate shown in their respective exponents; the viscous attenuation is more marked for the short capillary wave than for the gravity wave since $\kappa_2 > \kappa_1$. For $u_o > 1$, we may define an attenuation length for waves of wave number $k_m \kappa$ to be (in the physical units)

$$\ell = \frac{(u_o^4 - 1)^{1/2}}{4\alpha u_o^2 \kappa^2 k_m} = \left(\frac{c_m}{4\nu k_m^2}\right) \frac{(u_o^4 - 1)^{1/2}}{\kappa^2 u_o}$$

which corresponds to an attentuation factor e^{-1} at $x = \ell$. Now from (27),

$$u_o^2 = \frac{1}{2}(\kappa + \frac{1}{\kappa}) = \frac{1}{2}\left(\frac{\lambda}{\lambda_m} + \frac{\lambda_m}{\lambda}\right)$$

where $\lambda = \lambda_m/\kappa$ is the wave length and λ_m, the wave length at $\kappa = 1$. Then

$$\ell = \left(\frac{c_m}{4\nu k_m^2}\right)\left(\frac{\lambda}{\lambda_m}\right)^2 \left|\frac{\lambda}{\lambda_m} - \frac{\lambda_m}{\lambda}\right| \left[2\left(\frac{\lambda}{\lambda_m} + \frac{\lambda_m}{\lambda}\right)\right]^{-1/2} . \qquad (50)$$

This expression is probably accurate for $\lambda > 3\lambda_m$ or $< \frac{1}{3}\lambda_m$ (since H_p has a different expression near $u_o = 1$, or near $\lambda = \lambda_m$). For a water surface, $\lambda_m = 1.73$ cm. and $(c_m/4\nu k_m^2) = 44$ cm. Hence when $\lambda = \lambda_m/3$, $\ell = 5.05$ cm., $\lambda = 3\lambda_m$, $\ell = 410$ cm., and when $\lambda = 100\lambda_m$, $\ell = 3$ kilometers. Thus, the viscous damping on very long waves is hardly noticeable. Furthermore, it is of interest to note that for $u_o > 1$, the viscous effect only attenuates the wave amplitude without affecting, up to $O(\alpha)$, the wave length, its velocity or its phase, provided u_o is not too close to unity.

For $u_o < 1$, the attenuation of the wave is due to both the dispersion (given by the term $(1-u_o^4)^{1/2}$ in the exponent) and the viscosity; the former effect becomes more marked than the latter as the velocity u_o becomes smaller. Finally, when $u_o = 1$, the viscous effect then modifies slightly the wave length from the nonviscous value k_m to $\left[1 + 2(\nu k_m/c_m)^{1/2}\right] k_m$, and gives an attenuation length, in the physical units

$$\ell = \left[2 k_m \alpha^{1/2}\right]^{-1} = \left[c_m/(4\nu k_m^3)\right]^{1/2}$$

(which is equal to 3.48 cm. for a water surface). Furthermore, the viscosity now greatly affects the wave amplitude; in fact, the amplitude has no limit as $\alpha \to 0$. Therefore, to get an intelligible result in this case it is necessary to retain the viscosity. Or, what is equivalent, when viscosity is neglected, it has been shown in a previous work[6] that no steady state solution exists at $u_o = 1$.

The integral H_Q of (26) may be evaluated in a similar manner. With the terms of $O(\alpha)$ and higher orders neglected, one may verify that for $u_o > 1$,

$$H_Q = - \frac{u_o^2}{(u_o^4 - 1)^{1/2}} \exp\left[- \frac{4\alpha u_o^2 \kappa_1^2 x'}{(u_o^4-1)^{1/2}}\right] \left[\left(1 - \sqrt{2\alpha\kappa_1}\right) \cos \kappa_1 x' - \sqrt{2\alpha\kappa_1} \sin \kappa_1 x'\right]$$

$$+ M(x') \qquad \text{for} \quad x > 0 ,$$

$$= -\frac{u_o^2}{(u_o^4-1)^{1/2}} \exp\left[-\frac{4a u_o^2 \kappa_2^2 x'}{(u_o^4-1)^{1/2}}\right]\left[(1-\sqrt{2a\kappa_2})\cos\kappa_2 x' - \sqrt{2a\kappa_2}\sin\kappa_2 x'\right]$$

$$+ M(x') \qquad \text{for} \quad x<0 ,$$

where

$$M(x) = \frac{u_o^2}{\pi} R\ell \int_0^\infty \frac{e^{-|x|w}(1-2\sqrt{aw})}{1-2iu_o^2 w - w^2}\, dw ,$$

which is again significant only in a vicinity of the origin. The value of H_Q for $u_o \leq 1$ will, however, be omitted here.

5. Solution for Large Values of a (Small Reynolds Number)

When the parameter a is large, the evaluation of integral (24) becomes more complicated. However, when $|x|$ is also large, we expect that some method, such as Watson's lemma[8] or the principle of steepest descent[9], can be applied to obtain the asymptotic representation of the solution. Now, by changing the integration variable in (24) to t, with the function $k(t)$ given by (28), the integrand then contains the term $\exp(ix'k(t))$. Consequently the saddle points, determined from $dk/dt = 0$, are $t=i$ and $t=-i$. The path of steepest descent through these two saddle points is obtained from the conditions $\mathrm{Im}(ik(t)) = 0$ (since $\mathrm{Im}(ik)$ vanishes at both $t=i$ and $-i$) and $R\ell\,(ixk) \leq 0$. Let $t=\xi+i\eta$, then the steepest paths, determined from $\mathrm{Im}(ik) = 0$, are $\xi = 0$ and $\xi^2+\eta^2 = 1$. Hence it follows that for $x>0$, the path of steepest descent is from $\eta = 1$ to $\eta = +\infty$ along $\xi = 0$; and for $x<0$, it is from $\eta = 1$ along $\xi = +(1-\eta^2)^{1/2}$ to $\eta = -1$ and then to $\eta = -\infty$ along $\xi = 0$. These paths correspond respectively to the positive and negative imaginary k-axis.

Furthermore, it will be seen shortly that, even for small and moderate values of $|x|$, it is also of advantage to deform the integration path to the imaginary k-axis.

Because of the complexity displayed in the present case of large α, we shall, for simplicity, confine ourselves in the remaining part of this work to the case $u_o = 0(1)$ with $\alpha \gg 1$. We expect that the result of this case would also exhibit some typical features of the solution when both α and u_o are large. On the other hand, when α is large but $u_o = 0(\alpha^{-1/2})$ or less, the result reveals that the viscous effect is rather insignificant, primarily due to the small variations of the whole velocity field, and hence the inviscid solution becomes then a good approximation.

Now, for $x > 0$, we again choose the closed contour Γ to be the boundary of the first quadrant of the k-plane. Then, as has been shown, Γ will always enclose one simple pole at $k = k_1$ with k_1 given by (41)-(43) for $\alpha \gg 1$. Hence, by applying the theorem of residues to (24), we obtain for $x > 0$ again the same formal representation of H_P as given by (44) in which the quantities Res. (k_1) and $L(x)$ now have to be calculated differently for α large. The residue of the integrand at k_1 can be calculated from (44c) by making use of (25) and (42). In this manner, one obtains, after some manipulation,

$$\text{Res.}(k_1) = -\frac{i}{4u_o\sqrt{\alpha}} \exp\left\{-\frac{(1-i)x'}{2u_o\sqrt{2\alpha}} - i\frac{\pi}{4}\right\}\left[1 + 0(\alpha^{-1})\right] \quad \text{for } u_o = 0(1). \tag{51}$$

Next, the integral $L(x)$ of (44b) will be approximated for both large and small values of x'. If we introduce the new variable $s = -i\alpha k$ in (44b), then

$$L(x) = -\frac{u_o^2}{\pi a} \, R\ell \; i \int_0^\infty e^{-\frac{x'}{a}s} \left\{ 1 - \left(\frac{s}{a}\right)^2 - 2i\left(\frac{u_o^2}{a}\right)\left[s(1+2s)^2 \right.\right.$$

$$\left.\left. - 4s^{5/2}(s+1)^{1/2} \right]\right\}^{-1} ds \,. \qquad (52)$$

For $x'/a \gg 1$, the above integral representation is of the form to which Watson's lemma[8] can be readily applied. Since most of the contribution of the integral comes from a neighborhood of $s = 0$, we expand the integrand for small s and large a; then

$$L(x) \cong \frac{2u_o^4}{\pi a^2} \int_0^\infty e^{-\frac{x'}{a}s} \left[s + 4s^2 - 4s^{5/2} + \left(4 - \frac{u_o^4}{a^2}\right) s^3 - 2s^{7/2} + 0(s^4) \right] ds$$

$$\cong \frac{2u_o^4}{\pi a^2} \left\{ \left(\frac{a}{x'}\right)^2 + 8\left(\frac{a}{x'}\right)^3 - \frac{15}{2}\sqrt{\pi}\left(\frac{a}{x'}\right)^{7/2} + 0\left(\frac{a}{x'}\right)^4 \right\} \,, \qquad (53)$$

which is valid for $x' \gg a \gg 1$ and $u_o = 0(1)$. Thus, $L(x)$ is of the order a^{-2} and falls off like x^{-2} as $x \to +\infty$.

For small and moderate values of x'/a, we use (28) to obtain

$$L(x) = -\frac{\epsilon^2}{\pi} \, R\ell \; i \int_1^\infty \frac{\exp\left[-\frac{x'}{4a}(\eta-1)^2/\eta \right] \eta(\eta^2-1)\, d\eta}{\eta^3 - \left(\frac{1}{4a}\right)^2 \eta(\eta-1)^4 - i\epsilon^2(\eta-1)^2(\eta^3+\eta^2-\eta+1)} \,, \quad \left(\epsilon^2 = \frac{u_o^2}{4a}\right). \qquad (54)$$

Writing the exponential function above as

$$\exp\left[-\frac{x'}{4a} \frac{(\eta-1)^2}{\eta} \right] = \exp\left(-\frac{x'}{4a}\eta\right) \exp\left(\frac{x'}{2a}\right) \exp\left(-\frac{x'}{4a\eta}\right) \,,$$

the last factor may be expanded in a power series which converges rapidly, for $x'/4a$ small, inside the range of integration $\eta \geq 1$. To further simplify the calculation, the term with a^{-2} in the denominator may be neglected, for this process merely amounts to introducing a correction factor $[1+0(a^{-2})]$ to the final result. Furthermore; in order to retain the convergence of the integral at $x=0$, the terms η^3 and $i\epsilon^2\eta^3(\eta-1)^2$ in the denominator must be grouped together. Finally, we introduce the transformation $s=\epsilon(\eta-1)$ and expand the integrand for the small parameter $\epsilon = u_o/2\sqrt{a}$, giving

$$L(x) \cong \frac{\epsilon}{\pi} e^{\frac{x'}{2a}} R\ell \int_o^\infty e^{-\frac{x'}{4a\epsilon}s} \left[1-\frac{\epsilon x'}{4a}\frac{1}{(s+\epsilon)}+0\left(\frac{\epsilon x'}{a}\right)^2\right]\left[1-\frac{\epsilon^2}{(s+\epsilon)^2}\right]$$

$$\left[\frac{1}{s^2+i}-\frac{\epsilon s^3}{(s+\epsilon)^3(s+i)^2}+0(\epsilon^2)\right] ds.$$

If the term (s^2+i) is factorized as $(s+e^{-i\pi/4})(s-e^{-i\pi/4})$ and then the partial fraction is carried out, each term of the integrand can be integrated in terms of the exponential integral defined by (45a). From the known expansion of the exponential integral[9], one readily obtains

$$L(x) \cong \frac{u_o}{4}(2a)^{-1/2} - \frac{x'}{4\pi a}\left(1-\gamma_E-\log\frac{x'}{2u_o\sqrt{a}}\right)+0\left(\frac{x'^3}{a^2}\log\frac{x'^2}{a}, \frac{1}{a^2}\log a\right), \tag{55}$$

valid for $0 \leq x' < a^{1/2}$, $a \gg 1$ and $u_o = 0(1)$, ($\gamma_E = 0.5772\ldots$, Euler's constant).

To sum up, we combine (51) and (44), then for $a \gg 1$, $u_o = 0(1)$,

$$H_p(x) = -\frac{u_o}{2\sqrt{a}} e^{-\frac{x'}{2u_o\sqrt{2a}}} \cos\left(\frac{x'}{2u_o\sqrt{2a}}-\frac{\pi}{4}\right)\left[1+0(a^{-1})\right]+L(x) \quad \text{for } x>0 \tag{56}$$

where $L(x)$ is given by (54) and is approximated as given by (55) for $0 \leq x' < a^{1/2}$ and by (53) for $x' \gg a$.

Finally, for $x < 0$ and a large ($a > 1.0957$ to be precise), the path of integral (24) may be deformed to the negative imaginary k-axis since the integrand is then free from singularities inside the fourth quadrant of the k-plane. After using the transformation (28), we obtain in this case

$$H_P(x) = H_1(x) + H_2(x) = \frac{2\epsilon^2}{\pi} R\ell \; i \int_0^{\pi} \frac{\exp\left[\frac{x'}{2a}(1-\cos\theta)\right] \sin\theta \, d\theta}{1 + (1-\cos\theta)^2 \left[4\epsilon^2(\sin\theta + i\cos^2\theta) - (2a)^{-2}\right]}$$

$$+ \frac{\epsilon^2}{\pi} R\ell \; i \int_1^{\infty} \frac{\exp\left[\frac{x'}{4a}(\eta+1)^2/\eta\right] \eta(\eta^2-1)\, d\eta}{\eta^3 - (4a)^{-2}\eta(\eta+1)^4 - i\epsilon^2(\eta+1)^2\left[\eta^3 - \eta^2 - \eta - 1\right]},$$

$$(\epsilon^2 = \frac{u_o^2}{4a}), \qquad (57)$$

where the first integral H_1 comes from the integration along the right half of the unit circle $|t| = 1$, and H_2, from the integration along the imaginary axis, $t = -i\eta$, $\eta \geq 1$. If we expand the integrand of H_1 for small ϵ and large a, each term can be integrated exactly. By using $(1-\cos\theta) = 2t$, H_1 becomes

$$H_1(x) = \frac{4}{\pi}\left(\frac{u_o^4}{a^2}\right) \int_0^1 e^{\frac{x'}{a}t}\, t^2(1-2t)^2 \left[1 - 64\left(\frac{u_o^2}{a}\right) t^{5/2}(1-t)^{1/2}\right] dt \; \left\{1 + 0(a^{-2})\right\}$$

$$= \frac{8}{\pi}\left(\frac{u_o^4}{a^2}\right)\left\{x_o^{-3} - 12x_o^{-4} + 48x_o^{-5} - e^{-x_o}\left[\frac{1}{2}x_o^{-1} + 3x_o^{-2} + 13x_o^{-3} + 36x_o^{-4} + 48x_o^{-5}\right]\right\}$$

$$+ 0(a^{-3}) \qquad (58)$$

where $x_o = -x'/\alpha$. For $x' \ll -\alpha$, $H_2(x)$ is exponentially small, hence the value of $H(x)$ is well approximated by $H_1(x)$ alone, giving

$$H_P(x) \cong \frac{8}{\pi}\left(\frac{u_o^4}{\alpha^2}\right)\left\{\left(\frac{\alpha}{|x'|}\right)^3 - 12\left(\frac{\alpha}{x'}\right)^4 - 48\left(\frac{\alpha}{x'}\right)^5\right\} + 0(\alpha^{-3}) \quad \text{for} \quad x' \ll -\alpha. \tag{59}$$

On the other hand, for $-\alpha \ll x' \leq 0$, the value of H_1 may be obtained either from (57) or by expanding $\exp(x't/\alpha)$ in (57) into a series and integrating termwise, giving

$$H_1(x) \cong \frac{8}{15\pi}\left(\frac{u_o^2}{\alpha}\right)^2\left\{1 + \frac{7}{8}\left(\frac{x'}{\alpha}\right) + \frac{11}{28}\left(\frac{x'}{\alpha}\right)^2 + 0\left(\frac{|x'|^3}{\alpha^3}, \frac{1}{\alpha}\right)\right\}$$

which is of the order α^{-2}. The integral H_2 in (57) is similar in form to the integral in (54) and can be approximated for $-\alpha \ll x' < 0$ by using a method similar to that following (54). As can be readily verified, H_2 has here the expansion equal to the negative of the expression on the right side of (55), with x' replaced by its absolute value. Thus, for $-\alpha \ll x' < 0$, H_2 constitutes the most significant part of H_P (since H_1 is here of the order α^{-2}). Hence,

$$H_P(x) \cong -\frac{u_o}{4\sqrt{2\alpha}} - \frac{x'}{4\pi\alpha}\left(1 - \gamma_E - \log\frac{|x'|}{2u_o\sqrt{\alpha}}\right) + 0\left(\frac{|x'|^3}{\alpha^2}\log\frac{x'^2}{\alpha}, \frac{1}{\alpha^2}\log\alpha\right) \tag{60}$$

valid for $-\alpha^{1/2} < x' \leq 0$, $\alpha >> 1$ and $u_o = 0(1)$. To summarize, $H_P(x)$ is approximated as (60) for $-\alpha^{1/2} < x' \leq 0$ and as (59) for $x' \ll -\alpha$.

From the above result, namely, Eqs. (56), (59) and (60), one may note the following features of the solution:

(i) The surface elevation H_P is continuous everywhere, including the origin, but dH_P/dx has a logarithmic singularity at the origin. (This feature of course can also be seen from the original integral representation 24.)

(ii) Within the region $0 < x' \ll a$, there exists a train of waves of amplitude $(u_o/2a^{1/2})$ and wave length (in the physical units)

$$\lambda = 4\pi u_o (2a)^{1/2} k_m^{-1} = 4\pi(\nu U/g)^{1/2}$$

which is large for large a. This wave is gradually attenuated due to viscosity at the rate $\exp\left(-\frac{x'}{2u_o\sqrt{2a}}\right)$.

(iii) For $x' > a$, H_P falls off from $0(a^{-1/2})$ to $0(a^{-2})$ and diminishes like x^{-2} as $x \to +\infty$.

(iv) For $x' < 0$, H_P falls off monotonically from $0(a^{-1/2})$ to $0(a^{-2})$ and eventually dies out like x^{-3} as $x \to -\infty$.

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